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CELLULAR MAPS BETWEEN POLYHEDRA

BY

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ABSTRACT. A compact subset X of a polyhedron P is cellular in P if there is a pseudoisotopy of P shrinking precisely X to a point. A proper surjection $f: P \to Q$ is cellular if each point inverse of f is cellular in P. We give certain conditions under which cellular maps between polyhedra are approximable by homeomorphisms. An example of a cellular map which is not approximable by homeomorphisms is also given.

Introduction. The idea of generalizing the concept of cellularity to polyhedra was put forth by J. W. Cannon in [**4**]. He defined a compact subset X of a polyhedron P to be cellular in P if there is a pseudoisotopy of P shrinking precisely X to a point, and a proper surjection $f: P \to Q$ between polyhedra to be a cellular map if for each $y \in Q$, the set $f^{-1}(y)$ is cellular in P. Cannon conjectured that if one could show that a cellular map $f: P \to Q$, where P or Q is an n-manifold, $n \neq 4$, could be approximated by homeomorphisms, then one might be able to prove more general results involving more complicated polyhedra.

One of the basic techniques involved in working with polyhedral cellularity is the intrinsic stratification of polyhedra. It should be pointed out that we will be using a topological stratification and not a piecewise-linear stratification, as used by Aiken [**1**].

The first result which dealt with cellular maps between polyhedra which were not necessarily manifolds was that of Handel [**8**]. He was able to show that if $f: P \to Q$ is a cellular map with P and Q having no 4-dimensional stratum and for each $y \in Q$, both y and $f^{-1}(y)$ lie in the same dimensional stratum of P and Q respectively, then f is approximable by homeomorphisms. (Note that in general, a cellular set need not be contained in a single stratum.)

In [**9**], we were able to prove that if $f: P \to Q$ is a cellular map with either P or Q a generalized n-manifold, $n \neq 4$, or an n-manifold with boundary, $n \neq 4, 5$, then f is approximable. These conditions on P and Q were highly restrictive in the sense that either P or Q was required to have at most two different strata.

Later, we were able to show that if either P and Q had dimension at most 3, any cellular map $f: P \to Q$ is approximable by homeomorphisms. Again, the hypotheses were very restrictive, but the result encouraged us to believe that perhaps all cellular maps were approximable, with the possible requirement that the polyhedra have no

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four-dimensional stratum. This conjecture could be seen as a direct analogy to the results of Armentrout [3], Siebenmann [14], and Edwards [5] for cellular maps between n-manifolds, $n \neq 4$.

In this paper we give an approximation theorem for cellular maps between polyhedra for which the main restrictions come from homotopy conditions placed on the way the strata of the polyhedra fit together. As usual, we also require that the four-dimensional stratum of the range polyhedron be empty [9, 10]. We also require that each stratum in the domain have *some* piecewise-linear manifold structure. This is done only to allow the use of piecewise-linear engulfing.

Finally, an example of a cellular map between polyhedra is given which is not approximable by homeomorphisms. Thus, Question 4.9 of [11] is answered affirmatively. This example was pointed out to us by R. J. Daverman and we wish to thank him for allowing us to present it in this paper. He and J. J. Walsh have been especially helpful in the preparation of this paper.

1. Definitions and background. A *polyhedron* P is a subset of some Euclidean space $\mathbf{R}^n$ such that each point $b \in P$ has a neighborhood $N = bL$, the join of b and a compact subset L of P. Throughout, P and Q will denote polyhedra. A compact subset X of P is *cellular* in P if there is a pseudoisotopy H_t: $P \to P$ such that X is the only nondegenerate point preimage of H_1. A proper surjection f: $P \to Q$ is a *cellular map* if, for each $y \in Q$, $f^{-1}(y)$ is cellular in P.

The *intrinsic dimension* of a point x in P, denoted $I(x, P)$, is given by $I(x, P) = \max\{n \in \mathbf{Z} \mid$ there is an open embedding h: $\mathbf{R}^n \times cL \to P$ with L a compact polyhedron and $h(\{0\} \times c) = x\}$, where cL is the open cone on L. The *intrinsic n-skeleton* of P is $P^{(n)} = \{x \in P \mid I(x, P) \leq n\}$, and the *intrinsic n-stratum* of P is $P[n] = P^{(n)} - P^{(n-1)}$. Define the *depth of the stratification*, $D(P)$, to be given by $D(P) = \max\{i - j \mid P[i] \neq \varnothing$ and $P[j] \neq \varnothing\}$. It is crucial to note that P^n is a closed subpolyhedron of P and that $P[n]$ is a topological n-manifold. (See [9] for more details.)

We state here some of the results on cellular maps for later use.

THEOREM 1.1 [9]. *The following are equivalent*:

(1) *X is a cellular subset of P.*

(2) *The projection π: $P \to P/X$ is approximable by homeomorphisms.*

(3) *$X = \bigcap_{i=1}^{\infty} N_i$, where the N_i's are homeomorphic cellular neighborhoods with $\overline{N}_{i+1} \subset N_i$.*

A *cellular neighborhood* is an open set in P which is homeomorphic to $\mathbf{R}^n \times$ cL, the neighborhoods used to describe the intrinsic dimension of points in P.

THEOREM 1.2 [10]. *Let f: $P \to Q$ be a cellular map with $Q[4] = \varnothing$. Given a component B of a stratum $Q[n]$, there is a component A of $P[n]$ such that $f \mid \overline{A}$: $\overline{A} \to \overline{B}$ is a cellular map.*

THEOREM 1.3 [10]. *Let f: $P \to Q$ be a cellular map with $Q[4] = \varnothing$. Then $f_j = f \mid P^{(j)}$: $P^{(j)} \to Q^{(j)}$ is a cellular map and $P[n] \neq \varnothing$ if and only if $Q[n] \neq \varnothing$.*

THEOREM 1.4 [**9**, **10**]. *Let* $f: P \to Q$ *be a cellular map with*

(1) *either* $\dim P \leq 3$ *or* $\dim Q \leq 3$ *or*

(2) *either* P *or* Q *is a generalized n-manifold*, $n \neq 4$. *Then* f *is approximable by homeomorphisms.*

The last theorem to be given in this section is a consequence of Handel's proof of his approximation theorem [**8**] and the fact that f_l: $P[l] \to Q[l]$ is a cellular map.

THEOREM 1.5. *Let* f: $P \to Q$ *be a cellular map with* $Q[4] = \varnothing$. *Then* f *may be approximated by cellular maps* $\tilde{f}$ *such at* $\tilde{f}_l$ *is a homeomorphism.*

2. An engulfing theorem. We present here an engulfing theorem which will be used later to construct homeomorphisms with compact support in a stratum. The theorem given is in a sense a generalization of Stallings engulfing [**16**], and it has a radial engulfing analog given by Ancel in his notes on engulfing [**2**]. We will not give a rigorous proof of the theorem, but a short intuitive sketch of how the proof is based on Stalling's proof.

THEOREM 2.1. *Let* M^n *be a piecewise-linear n-manifold*, $n \geq 5$. *Suppose that for an integer* r, $0 \leq r \leq n-3$, *there exist open sets* $\tilde{V}_i$ *and* $\tilde{W}_{i+1}$, $-1 \leq i \leq r$, *such that* $\tilde{V}_{i+1} \subset \tilde{V}_i$, $\tilde{W}_{i+2} \subset \tilde{W}_{i+1} \subset \tilde{V}_i$, *and each i-complex in* $\tilde{V}_i$ *may be homotoped into* $\tilde{W}_i$ rel $\tilde{W}_{i+1}$ *by a homotopy in* $\tilde{V}_{i-1}$.

Then given a closed complex J *in* $\tilde{V}_r$ *with* $\dim J \leq n-3$ *and a subcomplex* L *of* J *in* $\tilde{W}_{r+1}$ *such that* $\mathrm{cl}(J-L)$ *is the polyhedron of a finite r-subcomplex* R *of* J, *there is a ambient isotopy* h_t: $M \to M$ *and a compact subset* E *of* $\tilde{V}_{-1}$ *such that* $h_1(\tilde{W}_0) \supset K$ *and* $h_t \mid (M-E) \cup L = \mathrm{id}$.

SKETCH OF PROOF. If $r=0$, then for each point $x \in \mathrm{cl}(J-L)$, we can find a path from x to $\tilde{W}_0$ which lies in $\tilde{V}_{-1}$. These paths may be chosen to miss $J-x$ for each x, and to be pairwise disjoint. The open set $\tilde{W}_0$ may now be stretched out along each of these paths to cover J.

Assume the theorem true for integers less than r. By hypothesis, there is a homotopy g_t: $J \to \tilde{V}_{r-1}$ such that $g_t \mid L = \mathrm{id}$ and $g_1(J) \subset \tilde{W}_r$. As in Stalling's proof, we may assume that g_t is a "nice" homotopy and that the only obstruction to using the set $g(J \times I)$ as a guide in pulling $\tilde{W}_0$ out to cover $J = g(J \times \{0\}) = g_0(J)$ is a singularity set of g_t and its shadow. This shadow set will be of dimension less than r, and it will lie in $\tilde{V}_{r-1}$. Hence our inductive assumption will allow us to engulf the shadow of the singularity set with $\tilde{W}_0$ keeping fixed any part of the r-skeleton of $g(J \times I)$ already in $\tilde{W}_0$. Thus we may engulf one simplex of the homotopy at a times, possibly uncovering only points in the interior of $(n-2)$-simplices which were previously covered, until we have engulfed $g(J \times \{0\}) = J$.

3. Extending homeomorphisms of a stratum. In order to be able to use the engulfing theorem, we must be able to extend an engulfing homeomorphism defined on a stratum to a homeomorphism of the entire polyhedron. In this section we show that a homeomorphism h: $P[n] \to P[n]$ that is isotopic to the identity with compact support may be extended to a homeomorphism $\tilde{h}$: $P \to P$ which is also isotopic to the identity with compact support.

LEMMA 3.1. *Let* h_t: $\mathbf{R}^n \to \mathbf{R}^n$ *be an isotopy with compact support. There is an isotopy* $\tilde{h}_t$: $\mathbf{R}^n \times \mathrm{cL} \to \mathbf{R}^n \times \mathrm{cL}$ *with compact support such that* $\tilde{h}_t \mid \mathbf{R}^n = h_t$.

PROOF. If we express the open cone cL as $(L \times [0, \infty))/(L \times \{0\})$, each point in $\mathbf{R}^n \times \mathrm{cL}$ has a representation (y, s, x) where $y \in \mathbf{R}^n$, $s \in [0, \infty)$, and $x \in L$. Define $\tilde{h}_t$ by

$$\tilde{h}_t(y, s, x) = \begin{cases} (h_{t-s}(y)\, s, x), & t \geqslant s, \\ (h_0(y), s, x), & t \leqslant s. \end{cases}$$

The next proposition was noted by Haver (see [7]) as a consequence of handle straightening techniques of Edwards and Kirby [**6**].

PROPOSITION 3.2 [7]. *If* M^n *is an n-manifold and* H: $M^n \times I \to M^n$ *is an ambient isotopy with compact support, then there is a finite collection of isotopies* H_i: $M^n \times [i/k, (i+1)/k] \to M^n$, $0 \leqslant i \leqslant k - 1$ *such that*

(1) *each* H_i *has compact support in the interior of an n-cell* $B_i \subset M$,
(2) $H_0(M^n \times \{0\}) = \mathrm{id}$,
(3) $H_i(M^n \times \{(i+1)/k\}) = H_{i+1}(M^n \times \{(i+1)/k\})$,
(4) $H_{k-1} \mid (M^n \times 1) = H \mid (M^n \times 1)$.

THEOREM 3.3. *Let* H: $P[n] \times [0, 1] \to P[n]$ *be an ambient isotopy with compact support E in* $P[n]$. *Then given a neighborhood U of* $P[n]$ *in P, there is an ambient isotopy* $\tilde{H}$: $P \times [0, 1] \to P$ *such that*

(1) $\tilde{H}$ *has compact support in U,*
(2) $\tilde{H} \mid (P[n] \times 1) = H \mid (P[n] \times 1)$.

PROOF. We first apply Proposition 3.2 to obtain a finite sequence of isotopies H_i: $P[n] \to P[n]$, each having compact support in an n-cell B_i in $P[n]$ such that $H_{k-1} \mid (P[n] \times 1) = H \mid (P[n] \times 1)$.

Given an open subset W of $P[n]$ which is homeomorphic to $\mathbf{R}^n$ and a neighborhood U of $\overline{W}$, there is an open set $\tilde{W}$ in P such that $\tilde{W} \cap P[n] = W$, $\tilde{W} \cong \mathbf{R}^n \times \mathrm{cL}$, and $\mathrm{cl}\, \tilde{W} \subset U$ [**8, 12**]. Thus for each B_i, we can find an open set U_i in P such that $U_i \cong \mathbf{R}^n \times \mathrm{cL}$, $U_i \cap P[n] = \mathrm{int}\, B_i$, and $\overline{U_i} \subset U$. We now apply Lemma 3.1 to get an extension $\tilde{H}_i$ of H_i to P which has compact support in U_i. The desired isotopy $\tilde{H}$ is then the composition $\tilde{H}_{k-1} \cdots \tilde{H}_1$: $P \to P$.

4. The approximation theorem. Before stating the main result, we recall that if $X \subset Y$, *X is i-LCC in Y* if for each $x \in X$ and neighborhood U_x of x, there is a neighborhood V_x of x such that each map of an i-sphere into $V_x - X$ is null homotopic in $U_x - X$.

THEOREM 4.1. *Let P and Q be polyhedra such that*

(1) $Q[4] = \varnothing$,
(2) *each* $P[n]$ *has a piecewise-linear triangulation,*
(3) *for each* $n \geqslant 5$, $\overline{P[n]} - P[n]$ *is i-LCC in* $\overline{P[n]}$, $i = 1\,2$,
(4) *for each* $n \geqslant 5$, $\overline{Q[n]} - Q[n]$ *is* 1*-LCC in* $\overline{Q[n]}$.

Then each cellular map f: $P \to Q$ *is approximable by homeomorphisms.*

PROOF. The proof will be by induction on $D(P)$, the depth of stratification of P. If $D(P) = 0$, it follows from Theorem 1.3 that $n \neq 4$. Hence f is approximable by homeomorphisms according to Theorem 1.4.

We therefore assume that $D(P) \geqslant 1$. It may also be assumed that $P[0] = \varnothing$ (Theorem 1.1 implies that any discrete collection of cellular sets may be shrunk out), and that f_l: $P[l] \to Q[l]$ is a homeomorphism by Theorem 1.5. The induction step will consist of showing that f can be approximated by a cellular map $\tilde{f}$: $P \to Q$ such that $\tilde{f}$ is 1-1 over $Q[l]$. We would then be able to apply the inductive hypothesis to approximate $\tilde{f} \mid P - P[l]$ by a homeomorphism which may be extended to $P[l]$ by $\tilde{f}_l$.

In order to complete the necessary step, it suffices to prove the following inductive statement and follow that with an application of the Bing shrinking criterion.

$I(j, k)$: For each k-dimensional subpolyhedron K of $P[l]$, ε: $P \to (0, 1)$, and neighborhood U of $f^{-1}(f(K))$, there is a homeomorphism h: $P \to P$ such that

(1) $h \mid (P - U) \cup P[l] = \text{id}$,

(2) for each $x \in K$, diam $h(f^{-1}(f(x))) \cap P^{(j)} < \varepsilon(x)$,

(3) for each $x \in P$, $d(fh(x), f(x)) < \varepsilon(x)$.

Note first that $I(l, k)$ is true for each k with $h = \text{id}$. Also, $I(j, 0)$ is true for each j since $f^{-1}(f(K))$ will consist of a discrete collection of cellular subsets of P. Given such a discrete collection, ε: $P \to (0, 1)$, and neighborhood U, one can find cellular neighborhoods N_x, $x \in K$, such that

(1) $N_x \supset f^{-1}f(x)$ for $x \in K$,

(2) $N_x \cap N_y = \varnothing$ for $x \neq y$,

(3) $\overline{N}_x \subset U$ for each x,

(4) diam $f(N_x) < \min\{\varepsilon(z) \mid z \in N_x\}$.

The cone structure on each N_x may now be employed to define a homeomorphism h: $P \to P$ with the desired properties.

The proof of the inductive statement can now be completed by showing that the statements $I(j - 1, k)$ and $I(j, k - 1)$ imply $I(j, k)$.

Let K, U, and ε: $P \to (0, 1)$ be given. Triangulate K in such a way that for each k-simplex σ in K, diam $\sigma < \delta = \min\{\frac{1}{4} \in (x) \mid x \in \sigma\}$. It follows from $I(j, k - 1)$ that we may assume that, for each $x \in \partial\sigma$, diam $f_j^{-1}f_j(x) < \delta$ and from $I(j - 1, k)$ that, for each $x \in \sigma$, diam $f_{j-1}^{-1}f_{j-1}(x) < \delta$. Let $N_\sigma \cong \mathbf{R}^l \times \text{cL}$ be a neighborhood of $f_j(\text{int}\,\sigma)$ in $Q^{(j)}$ such that

(1) $N_\sigma \subset f_j(U \cap P^{(j)})$,

(2) diam $N_\sigma < \delta$,

(3) $f_j^{-1}(N_\sigma) \cap (K - \sigma) = \varnothing$,

(4) if we identify N_σ with $\mathbf{R}^l \times \text{cL}$, there is an l-cell B in $\mathbf{R}^l$ such that, for $y \in \mathbf{R}^l - \text{int}\, B$, diam$(f_j^{-1}(y \times \text{cL})) < \delta/4$.

Now $U_\sigma = f_j^{-1}(N_\sigma)$ is a neighborhood of $f_j^{-1}f_j(\text{int}\,\sigma)$ which lies in U.

Assume that $l < j \leqslant 3$. We may use Theorem 1.4 to approximate $f_j \mid U_\sigma$: $U_\sigma \to N_\sigma$ by a homeomorphism h_σ such that, for $y \in \mathbf{R}^l - \text{int}\, B$, diam $h_\sigma^{-1}(y \times \text{cL}) < \delta/4$. Let $A = h_\sigma^{-1}(B)$.

In [10], the possible structures of cellular neighborhoods $\mathbf{R}^l \times \text{cL}$ of dimension at most three were described in detail. We will not repeat that analysis here, but we will describe the various situations and refer the reader to that paper for further details.

In the case where $l = 1$ and $j = 2$, $\mathbf{R}^1 \times \mathrm{cL} \cong U_\sigma$ is homeomorphic to a finite number of copies of $\mathbf{R}^2_+ \cong \mathbf{R}^1 \times [0, \infty)$ identified along their common $\mathbf{R}^1$ boundary. In this situation we can find an isotopy h_t with compact support in each $A \times [0, \infty)$ which shrinks each $f_2^{-1}f_2(x)$ to have diameter less than $\varepsilon(x)$ for each $x \in U_\sigma \cap P[1]$ and is the identity on $\partial A \times [0, \infty) \cup A \times \{0\}$. The homeomorphism h_1 can be extended to $\tilde{H}_\sigma\colon P \to P$ with support in U in such a way that for $x \in K - f_2^{-1}f_2(\mathrm{int}\,\sigma)$, $f^{-1}f(x)$ is not moved. Piecing together these homeomorphisms, one for each k-simplex σ in K, yields the desired homeomorphisms. A similar argument holds for $l = 2$ and $j = 3$.

In the case where $l = 1$ and $j = 3$, we have by assumption that, for each $x \in \sigma$, $f_2^{-1}f_2(x) < \delta$. Thus we need only consider the 3-dimensional stratum of $\mathbf{R}^1 \times \mathrm{cL}$. It was shown in **[10]** that the 3-stratum can be dealt with by considering the set $\mathbf{R}^1 \times c(I^1)$ or $\mathbf{R}^1 \times c(S^1)$, with each $\mathbf{R}^1 \times c$ corresponding to the $\mathbf{R}^1 \times c$ in $\mathbf{R}^1 \times \mathrm{cL}$.

For $\mathbf{R}^1 \times c(S^1)$, one may proceed as in the above cases to find an isotopy with compact support in $(\mathrm{int}\,A \times cS^1) - (\mathrm{int}\,A \times c)$ which shrinks $f_3^{-1}f_3(x)$ to have diameter less than $\varepsilon(x)$. Again, we extend the final stage of the isotopy to all of P.

When considering $\mathbf{R}^1 \times c(I^1)$, we also identify a 1-cell D in $\mathrm{int}\,I^1$ such that, for each $x \in \mathrm{int}\,A$ such that $\mathrm{diam}\, f_3^{-1}f_3(x) > \varepsilon(x)$, $f_3^{-1}f_3(x) \subset \mathrm{int}\,A \times c(\mathrm{int}\,D)$. There is an isotopy of $\mathbf{R}^1 \times c(I^1)$ with compact support in $(\mathrm{int}\,A) \times c(\mathrm{int}\,D) - (\mathrm{int}\,A) \times c$ which shrinks each $f_3^{-1}f_3(x)$ to small size. The final stage can once more be extended to P yielding the desired result.

The hypothesis that $Q[4] = \varnothing$ and Theorem 1.3 together imply that $P[4] = \varnothing$. Therefore the statement $I(3, k)$ implies $I(4, k)$.

We now consider $I(j, k)$ for $j \geq 5$. Let N_σ and U_σ be as above, and let $U_j = U_\sigma \cap P[j]$. We can assume that U_j has only one component; otherwise we would work in one component of U_j at a time. In order to shrink each $f_j^{-1}f_j(x)$ which intersects U_j, $x \in \mathrm{int}\,\sigma$, we will need two engulfing lemmas. The lemmas will be stated, the proof of $I(j, k)$ completed, and then the proofs of the two lemmas given.

Let $X = f^{-1}f(\sigma) \cap U_j$ and $C = [f^{-1}f(\sigma - \mathrm{int}\,A)] \cap U_j$.

LEMMA 1. *There exist an open set V in U_j with $X \subset V$ and open sets W and N in U_j for which*

(1) $\overline{W} \subset N \subset \overline{N} \subset V$ *(closures taken in V)*,

(2) $N \subset N_\delta(\sigma; P) \cap U_j$,

(3) $C \subset W$,

such that for each polyhedron J in V of dimension at most $n - 3$ with subpolyhedron $L \subset N$ and $\overline{J - L}$ compact, there is a homeomorphism $h_1\colon U_j \to U_j$ which is isotopic to the identity with compact support in $U_j - \mathrm{cl}\,W$ such that $h_1(N) \supset J$.

LEMMA 2. *Let V be the open set given in Lemma* 1. *For each polyhedron J in V with* $\dim J \leq 2$ *and subpolyhedron L of J for which $L \subset V - X$ and $\overline{J - L}$ is compact, there is a homeomorphism $h_2\colon V \to V$ isotopic to the identity with compact support in V such that $h_2(V - X) \supset J$.*

We now complete the proof of $I(j, k)$. Let S be a triangulation of V, and let R be the 2-skeleton of $\mathrm{st}(V - W, S)$. An application of Lemma 2 yields a homeomorphism h_2: $V \to V$ such that $h_2(V - X) \supset R$ and h_2 is isotopic to the identity with support in a compact subset E of V. Define $J^* = \{\tau \in S \mid \tau \subset \overline{W} \cup (V - (E \cup X))\}$ and let J be the dual codimension-three polyhedron in V. There is a homeomorphism h_1: $U_j \to U_j$ provided by Lemma 1 such that $h_1(N) \supset J$ and h_1 have compact support in $U_j - \mathrm{cl}W$. We extend h_1 by the identity to U_j and then use the join structure between R and J in V to obtain a homeomorphism h_3: $U_j \to U_j$ such that $h_1(N) \cup h_3 h_2(U_j - X) = U_j$. The homeomorphisms $h_\sigma = h_1^{-1} h_3 h_2$ may be extended to all of P in such a way that no points of $f^{-1}f(K - \mathrm{int}\,\sigma) \cup P - U$ are moved, and $h_\sigma(X) \subset N$. The proof of $I(j, k)$ is completed by composing homeomorphisms defined as above for each k-simplex σ in K.

PROOF OF LEMMA 1. The proof will consist of constructing neighborhoods V_i of $f_j^{-1}f_j(\mathrm{int}\,\sigma)$ in $U_\sigma \cap P^{(j)}$, and neighborhoods W_i of $\mathrm{int}\,\sigma$ in $U_\sigma \cap P^{(j)}$ such that $V_i \subset V_{i-1}$, $W_{i+1} \subset W_i \subset N_\delta(\mathrm{int}\,\sigma; P)$, $W_{i+1} \subset V_i$, and a neighborhood W of $\mathrm{int}\,\sigma \cup f_j^{-1}f_j(\sigma - \mathrm{int}\,A)$ with $W \subset W_i$ such that $V_i - \overline{W}$ may be homotoped stratum preserving into $W_i - \overline{W} \,\mathrm{rel}\, W_{i+1}$ in $V_{i-1} - \overline{W}$ for $0 \leq i \leq j - 3$. We will then apply Theorem 2.1 to the manifold $U_j - \overline{W}$ with $\tilde{V}_i = (V_i - \overline{W}) \cap P[j]$ and $\tilde{W}_i = (W_i - \overline{W}) \cap P[j]$. The desired set V is $\tilde{V}_{j-3}$.

The homotopies and neighborhoods will be constructed in the same manner as those of Proposition 2.2 of [**10**]. Choose a neighborhood R of $\mathrm{int}\,\sigma$ in $U_\sigma \cap P^{(j)}$ such that

(1) for each $x \in \mathrm{int}\,\sigma - \mathrm{int}\,A, f_j^{-1}f_j(x) \subset R$,

(2) $R \subset N_\delta(\sigma; P)$.

Cover $f_j^{-1}f_j(\mathrm{int}\,\sigma)$ with a locally finite collection of saturated open sets $\{N_\alpha^k\}$ such that

(1) $\cup N_\alpha^k \subset U_\sigma$,

(2) for each N_α^k, there is a cellular neighborhood C_α^k of the form $\mathbf{R}^l \times \mathrm{cL}$ such that $\overline{N}_\alpha^k \subset C_\alpha^k \subset \overline{C}_\alpha^k \subset U_\sigma$,

(3) for each C_α^k, if $C_\alpha^k \cap (\mathrm{int}\,\sigma - \mathrm{int}\,A) = \varnothing$, then $\overline{C}_\alpha^k \subset R$.

Let T_k be a triangulation of $\mathrm{int}\,\sigma$ such that $\mathrm{bd}\,A$ is a subcomplex and that, for each $\tau \in T_k$, $f_j^{-1}f_j(\tau)$ lies in some U_α^k. Then for each $(k-1)$-simplex γ in the $(k-1)$-skeleton T_k^{k-1} of T_k, cover $f_j^{-1}f_j(\gamma)$ by a finite number of saturated open sets $\{U_\beta^{k-1}\}$ such that for each U_β^{k-1} there is a cellular neighborhood $C_\beta^{k-1} \cong \mathbf{R}^l \times \mathrm{cL}$ for which $f_j^{-1}f_j(\gamma) \subset U_\alpha^k$ implies that $\overline{U}_\beta^{k-1} \subset C_\beta^{k-1} \subset \overline{C}_\beta^{k-1} \subset U_\alpha^k$. T_{k-1} will then be a subdivision of T_k^{k-1} such that, for each simplex $\tau \in T_{k-1}, f_j^{-1}f_j(\tau) \subset U_\beta^{k-1}$ for some β. Similarly define T_{n-1}, $\{U_\beta^{n-1}\}$, and $\{C_\beta^{n-1}\}$ given T_n, $\{U_\alpha^n\}$, and $\{C_\alpha^n\}$. We also require that $\{C_\beta^0\}$ be a pairwise disjoint collection of cellular neighborhoods.

We will define $V_{-1} = U_j$, and now describe V_0 and W_0. Let $A_0 = \cup\{U_\beta^0\}$, and for each 1-simplex $\tau \in T_1$, define A_τ to be a saturated open set containing $f_j^{-1}f_j(\tau - A_0)$ which lies in some U_α^1. We also require that if τ and γ are different 1-simplices in T_1, then $\overline{A}_\tau \cap \overline{A}_\gamma = \varnothing$. Define $A_1 = \cup\{A_\tau \mid \tau \text{ is a simplex in } T_1\}$. Similarly construct A_n, $1 \leq n \leq k$, and let $V_0 = \cup_{n=0}^k A_n$.

The desired homotopy will first pull A_0 into R in such a way that those neighborhoods A_τ of 0-simplices τ which lie in $\operatorname{int}\sigma - \operatorname{int} A$ are fixed, while the other neighborhoods used to define A_0 are pulled close to $P[l]$ using the cone structure on the C_β^0's. This may be done so that the subsequent homotopies will not move the image of A_0, and so that no points are moved in a neighborhood of $P[l] \cup f_j^{-1}f_j(\operatorname{int}\sigma - \operatorname{int} D_1)$, where D_1 is a k-cell contained in A. Let H_t^0: $V_0 \to V_{-1}$ be this homotopy. In the same manner, construct homotopies H_t^n: $H_1^{n-1} \cdots H_1^0(V_0) \to V_{-1}$ such that $H_t^n \mid H_1^{n-1} \cdots H_1^0(\bigcup_{j<n} A_j) = \mathrm{id}$, $H_1^n(\bigcup_{j=0}^n A_j) \subset R$, and H_t^n moves no points in a neighborhood of $P[l] \cup f_j^{-1}f_j(\operatorname{int}\sigma - \operatorname{int} D_n)$, where D_n is a k-cell contained in D_{n-1}. The desired homotopy G is then $G = H^kH^{k-1} \cdots H^0$: $V_0 \to V_{-1}$. Let W_0 be a neighborhood of $G_1(V_0)$ which lies in R. Note also that there is a neighborhood W_1^* of $f_j^{-1}f_j(D_k) \cup P[l]$ which lies in $G_1(V_0)$ on which G_t is the identity. We now repeat the above construction with V_{-1} replaced by V_0, W_1^* replacing R, and identify $V_1 \subset V_0$, $W_1 \subset W_0$, and $W_2^* \subset W_1$. The sets V_n, W_n, and W_{n+1}^* are thus defined inductively. Finally, W is a neighborhood of $P[l] \cup f_j^{-1}f_j(\operatorname{int}\sigma - \operatorname{int} A)$ whose closure lies in W_{j-1}^*.

We note here for later use that we could have modified this procedure to describe a neighborhood D of $f_j^{-1}f_j(\operatorname{int}\sigma)$ in V_{j-3} and a homotopy h_t: $D \to V_{j-3}$ such that for each $x \in D$ there is $s_x \in [0, 1)$ for which $I(h_t(x), P) = I(x, P)$ for $t < s_x$ and $I(h_t(x), P) = l$ for $t \geqslant s_x$. The changes required would be to cover all of $f_j^{-1}f_j(\operatorname{int}\sigma)$ and have the homotopies use the cellular neighborhoods to pull each A_i *into* $P[l]$ keeping $A_i \cap P[l]$ fixed.

PROOF OF LEMMA 2. We wish to apply Stallings' engulfing theorem [**16**] to the manifold V. Thus we must show that $(V, V - X)$ is 2-connected.

Let x be a point in X. Since $V_{j-3} \supset f_j^{-1}f_j(\operatorname{int}\sigma)$, there is a cellular neighborhood U_x containing $f_j^{-1}f_j(x)$ with $U_x \subset V_{j-3}$. Since $f_j(U_x) \not\subset Q^{(j-1)}$ by Theorem 1.2, there is a point $y \in (U_x - f_j^{-1}f_j(\operatorname{int}\sigma)) \cap P[j]$. It now follows that there is a path from x to y in $U_x \cap P[j]$ and $(V, V - X)$ is 0-connected.

Suppose that g: $(B^1, \partial B^1) \to (V, V - X)$ is given, where B^1 is a 1-cell. As was noted before, there is a neighborhood D of $f_j^{-1}f_j(\operatorname{int}\sigma)$ in V_{j-3} and a homotopy h_t: $D \to V_{j-3}$ such that, for each $x \in D$, $I(h_t(x), P) = I(x, P)$ for $t < s_x$ and $I(h_t(x), P) = l$ for $t > s_x$. Let $\tau_1, \tau_2, \ldots, \tau_n$ be disjoint arcs in B^1 such that

(1) $g(\tau_i) \cap X \subset g(\operatorname{int}\tau_i)$,

(2) $g(B^1) \cap X \subset \bigcup_{i=1}^n g(\operatorname{int}\tau_i) \subset \bigcup_{i=1}^n g(\tau_i) \subset D \cap P[j]$.

Given a component U of $D \cap P[j]$, $U - f_j^{-1}f_j(\sigma)$ is path connected (see Proposition 2.3 of [**10**]). Hence we may connect the points of $g(\partial\tau_i)$ by a path α_i: $\tau_i \to (D \cap P[j]) - X$. Since D may be homotoped to $D \cap P[l]$ in a stratum respecting manner, the singular loop $\alpha_i(\tau_i) \cup g(\tau_i)$ bounds a singular disk in $V \cup (D \cap P[l])$. Since $V_{j-3} \cap P[l]$ is homeomorphic to $\mathbf{R}^l$ and hence has a product neighborhood $\mathbf{R}^l \times \mathrm{cL}$ in V_{j-3}, we can assume that the singular disk intersects $P[l]$ in a single point in V_{j-3}. That point may then be eliminated by the 1-LCC hypothesis in P. Now α_i is homotopic to $g \mid \tau_i$ in V, and it follows that $(V, V - X)$ is 1-connected.

It remains to be shown that $(V, V - X)$ is 2-connected. If g: $(B^2, \partial B^2) \to (V, V - X)$ is given, choose disjoint disks-with-holes $B_1, B_2, \ldots, B_n$ such that

(1) $g(B_i) \cap X \subset g(\text{int } B_i)$,
(2) $g(B^2) \cap X \subset \bigcup_{i=1}^{n} g(\text{int } B_i)$,
(3) $g(B_i) \subset D \cap P[j]$,
where D is the neighborhood described above.

We first want to approximate $fg \mid B_i$: $B_i \to N_\sigma$ by a map k_i: $B_i \to N_\sigma$ such that $k_i \mid \partial B_i = fg \mid \partial B_i$ and $k_i(\text{int } B_i) \subset N_\sigma \cap Q[j] \cap f_j(D)$. For each point $y \in fg(B_i) \cap Q[k]$, let N_y be a neighborhood of y homeomorphic to $\mathbf{R}^k \times \text{cL}_k$ which lies in $f_j(D)$. Cover $fg(B_i)$ by finitely many such neighborhoods $N_1, \ldots, N_m$ and let T be a triangulation of int B_i such that, for each simplex τ of T, $\text{diam}\, \tau < d(\tau, \partial B_i)$ and each $fg(\tau)$ lies in some N_s. For each point y in the 0-skeleton of T with $fg(y) \in Q[k]$, we will approximate $fg(y)$ by a point in the component of $N_y \cap Q[j]$ which corresponds to the component of $f_j^{-1}(N_y) \cap P[j]$ containing $g(y)$. (See Theorem 1.2.) This is done so that the new map k_i defined on the 0-skeleton of T sends each point into $Q[j] \cap N_\sigma$ so close to its image under fg that, for each 1-simplex τ with $fg(\tau) \subset N_s$, $k_i(\partial \tau)$ again lies in N_s. We may connect the points $k_i(\partial \tau)$ in $N_s \cap Q[j]$ by an arc whose diameter is less than twice the diameter of $fg(\tau)$. This arc will then be $k_i(\tau)$. Now for each 2-simplex γ in T, $k_i(\partial\gamma)$ is already defined, and lies in some $N_s \cap Q[j]$. This singular 1-sphere bounds a singular 2-cell in $(N_s \cap Q[j]) \cup (N_s \cap Q[k])$, when $N_s \cong \mathbf{R}^k \times \text{cL}_k$. Again we may assume that this singular 2-cell intersects $Q[k]$ in a single point and use the 1-LCC condition on Q to cut this singular 2-cell off in $Q[j]$. We now have a map k_i: int $B_i \to Q[j] \cap f(D)$ which may be extended to ∂B_i by the map fg. Thus if we define k: $B^2 \to N_\sigma$ to be k_i on B_i and fg on $B^2 - \bigcup_{i=1}^{n} B_i$, we have a map of B^2 into $N_\sigma - f_j(\sigma)$.

An approximate lift $\tilde{k}$: $B^2 \to V - X$ of the map k: $B^2 \to N_\sigma - f_j(\sigma)$ is constructed so that, for each i, $\tilde{k}(B_i) \subset (D \cap P[j]) - X$ and $\tilde{k} \mid \partial B_i = g \mid \partial B_i$. Cover $k(B_i)$ with cellular neighborhoods to get open covers $V_0 = \{V_{01}, V_{02}, \ldots, V_{0n}\}$, $V_1 = \{V_{1,1}, \ldots, V_{1,m}\}$, and $V_2 = \{V_{2,1}, \ldots, V_{2,p}\}$ of $k(B_i)$ such that
(1) $\text{st}^2(k(y), V_r) \subset V_{r+1,s}$ for some s, where $y \in B_i$,
(2) $f_j^{-1}(V_{2,k})$ is contained in a cellular neighborhood in $D - f_j^{-1} f_j(\sigma)$, $1 \leqslant k \leqslant p$.

Triangulate B_i so that for each simplex τ of the triangulation, $k(\tau) \subset V_{0,s}$ for some s. Define the approximate lift on the 0-skeleton so that, for each point $y \in \partial B_i$, $\tilde{k}(y) = g(y)$. Given a 2-simplex γ of the triangulation and neighborhood $V_{0,s}$ containing $k(\gamma)$, we also require that the vertices of γ be mapped by $\tilde{k}$ into the same component of $f_j^{-1}(V_{0,s}) \cap P[j]$. Thus for each 1-simplex τ of the triangulation, define $\tilde{k}(\tau)$ to be a path between the points of $\tilde{k}(\partial\tau)$ which lies in $f_j^{-1}(V_{1,r}) \cap P[j]$, where $V_{1,r} \supset k(\tau)$. Again we require that, for $\tau \subset \partial B_i$, $\tilde{k}(\tau) = g(\tau)$.

For each 2-simplex γ in the triangulation, we have $\tilde{k}(\partial\gamma) \subset N_\gamma \cap P[j]$, where N_γ is a cellular neighborhood lying in D and containing $f_j^{-1}(V_{2,s}) \supset f_j^{-1}(k(\gamma))$. Hence $\tilde{k}(\partial\gamma)$ bounds a singular 2-cell in $(N_\gamma \cap P[j]) \cup (N_\gamma \cap P[n])$ where $N_\gamma \cong \mathbf{R}^n \times \text{cL}_n$. We again take this singular 2-cell so that it intersects $P[n]$ in a single point, and eliminate that point using the 1-LCC hypothesis on P. We now have a map $\tilde{k}$: $B_i \to (D \cap P[j]) - X$ which agrees with g on ∂B_i. Because $g(B_i) \cup \tilde{k}(B_i) \subset D$, $g(B_i)$ is homotopic to $\tilde{k}(B_i)$rel ∂B_i in $V \cup (D \cap P[l])$. As before, we can make the homotopy intersect $P[l]$ in a single point and then use the 2-LCC condition on P to cut the homotopy off of $P[l]$ in $P[j]$.

Thus for each i, we have a map $\tilde{k} \mid B_i\colon B_i \to V - X$ such that $\tilde{k} \mid \partial B_i = g \mid \partial B_i$ and $\tilde{k} \mid B_i$ is homotopic to $g \mid B_i$ rel ∂B_i in V. If we extend $\tilde{k}$ to $B^2 - \cup_{i=1}^n B_i$ by the map g, we then have a map $\tilde{k}\colon B^2 \to V - X$ which is homotopic to g rel ∂B_i in V.

5. Examples. Recently, R. J. Daverman pointed out the following example of a cellular map between polyhedra which is not approximable by homeomorphisms.

Let W be a contractible $(n + 1)$-manifold whose boundary is a nonsimply connected homology n-sphere H^n, $n \geq 4$, and choose $w_0 \in W$. The polyhedron P is then given by

$$P = (W \times S^1) \underset{w_0 \times S^1}{\cup} (W \times S^1),$$

two copies of $W \times S^1$ identified along the circle $w_0 \times S^1$ by the identity map. Choose $\tilde{W}$ to be a submanifold of W containing w_0 such that $W - \operatorname{int} \tilde{W} = H \times [0, 1]$. The polyhedron Q is defined by

$$Q = (\mathrm{cH} \times S^1) \underset{c \times S^1}{\cup} (\mathrm{cH} \times S^1),$$

with cH being the standard cone on the homology sphere H. The map to be considered is $f\colon P \to Q$ which takes each $(W \times y) \cup_{w_0 \times y} (W \times y)$ onto $(\mathrm{cH} \times y) \cup_{c \times y} (\mathrm{cH} \times y)$ by sending $(\tilde{W} \times y) \cup_{w_0 \times y} (\tilde{W} \times y)$ to the point $c \times y$.

We first note that $\tilde{W}$ is a cell-like subset of W, and hence that $\tilde{W} \times y$ is a cellular subset of $W \times S^1$. Since $w_0 \times S^1$ is a tame 1-sphere in $W \times S^1$, given a neighborhood U of $\tilde{W} \times y$ in $W \times S^1$, there is an open neighborhood V of $\tilde{W} \times y$ in $W \times S^1$ such that $\bar{V} \subset U$ and a homeomorphism $h\colon V \to \mathbf{R}^{n+2}$ which takes $(w_0 \times S^1) \cap V$ onto $\mathbf{R}^1 \times \{0\} \subset \mathbf{R}^{n+2}$. It now follows from Theorem 1.1 that $\tilde{W} \times y \cup_{w_0 \times y} \tilde{W} \times y$ is cellular in P.

In order to show that f is not approximable by homeomorphisms, it suffices to show that $P[1] = w_0 \times S^1$ is 1-LCC in P, but $Q[1] = c \times S^1$ is not 1-LCC in Q.

Given a point $x \in w_0 \times S^1$ and a neighborhood U of x in P, there is a neighborhood V of x in P such that $x \in V \subset U$ and $V \cong (W \times \mathbf{R}^1) \cup_{w_0 \times \mathbf{R}^1} (W \times \mathbf{R}^1)$. Each loop in $V - P[1]$ can be contracted in V missing $w_0 \times \mathbf{R}^1$ since $w_0 \times \mathbf{R}^1$ is tame in each of the contractible manifolds $W \times \mathbf{R}^1$.

On the other hand, given a neighborhood U of a point $y \in Q[1] \cong c \times S^1$, there is a neighborhood V of y in U such that $V \cong (\mathrm{cH} \times \mathbf{R}^1) \cup_{c \times \mathbf{R}^1} (\mathrm{cH} \times \mathbf{R}^1)$. Since $\pi_1(H) \neq 0$, there is a loop in V which does not bound a singular disk in $Q - Q[1]$. Hence $Q[1]$ is not 1-LCC in Q.

Another interesting feature of cellular sets in polyhedra is illustrated by a similar example. Let W and H be as before, and let $\tilde{P} = W \cup_H \mathrm{cH}$, the space obtained by identifying W and cH along the boundary H of W and the copy of H being coned over. The polyhedron P will be $(\tilde{P} \times S^1) \cup_{c \times S^1} (\tilde{P} \times S^1)$. If we let $\tilde{X} = \tilde{P} - \operatorname{int} B^{n+1}$, where B^{n+1} is a tame $(n + 1)$-cell in $\operatorname{int} W$, then $\tilde{X}$ is a contractible subset of $\tilde{P}$ and, for each $t \in S^1$, $\tilde{X} \times t$ is cellular in the manifold $\tilde{P} \times S^1$. However, $X = \tilde{X} \times t$ is not cellular in P. If X were cellular in P, then given a neighborhood N

of $c \times t$ which is homeomorphic to $(\mathrm{cH} \times \mathbf{R}^1) \cup_{c \times \mathbf{R}^1} (\mathrm{cH} \times \mathbf{R}^1)$, and a neighborhood U of X in P such that $U \cap P[1] \subset N$, there would be a contraction h_t of X in U such that, for each $x \in X$, $I(h_t(x), P) = I(x, P)$ for $t < 1$ and $I(h(x), P) = 1$ [**9**]. Suppose there were such a contraction. There are loops in $X \cap N$ arbitrarily close to $P[1]$ which do not bound in $N \cap P[n+2]$. However, all of these loops do bound in X. If the loop α in X is chosen close enough to $P[1]$, $h_t(\alpha) \subset N$ for $0 \leqslant t \leqslant 1$. But for some $s > 0$, $h_s(D) \subset N$, where D is a singular disk bounded by α in X. Hence every loop sufficiently close to $P[1]$ must bound in N, a contradiction.

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